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First-passage time and the fluctuation of the quenched disorder in biased media

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We consider the exit-time problem for a particle diffusing in a one-dimensional random medium with a global nonrandom bias field. It is found that, for small bias, in addition to the usual renormalization of the diffusion coefficient the fluctuations in the hopping rates have the effect of increasing the mean exit-time, independently of the direction of the bias and the initial position of the particle. We also introduce an effective-medium approximation that becomes exact in the limit of small bias.

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In the last decade, the escape time from a finite domain of a particle diffusing in a random medium has been the subject of considerable theoretical work [1]. Particularly, in a finite nonbiased random medium (without perfect traps) there always exists a nonzero escape probability, however the mean first-passage time (MFPT) to leave this finite domain can be a divergent quantity if the disorder is strong [2]. The study of the MFPT brings information about the long-time decay law of the survival probability, so by looking at the MFPT we can characterize the degree of disorder. In this Rapid Communication we consider a random walk (RW) on a finite one-dimensional disordered chain with hopping probabilities of the site-disorder type [1, 3] in the presence of a global nonrandom bias. We will consider only the case of weak disorder [the mean value of the inverse moments of the random hopping transitions w: $\beta_k \equiv \langle (1/w)^k \rangle$ are all finite quantities], leaving the analysis of biased anomalous diffusion (strong disorder) to be published elsewhere. The basic effect of weak disorder on an unbiased RW is to introduce an effective time scale through the occurrence of an effective diffusion coefficient (β_1^{-1}) [3]. The fluctuation of the quenched disorder (FQD) measured by

$$\mathcal{F} = \frac{\beta_2 - {\beta_1}^2}{{\beta_1}^2} \qquad (1)$$

does not enter into the expression for the MFPT in the absence of bias [2]. Here we show that a global bias makes the MFPT sensible to FQD, so that this last quantity becomes physically more relevant for the characterization of the temporal evolution of a biased RW on finite domains. In particular, FQD appears at least for small bias.

The presence of bias in a system introduces a substantial change in its dynamics. One classical model is the problem of diffusion in a medium with randomly distributed perfect traps. When a drift term is added to the diffusion equation, the asymptotic long-time structure of the survival fraction of particles is modified however small the bias field is [4]. Several results have been obtained concerning the survival probability (or related quantities) of a particle which performs a RW on a one-dimensional biased random medium. However, the main results in the literature, regarding the survival probability or MFPT, deal with only two models: media with randomly distributed perfect traps [5] or the Sinai model [6, 7]. Trapping in one dimension is a model of (very) strong disorder but allows the deduction of exact results with large bias. The Sinai model is a time discrete random walk on a one-dimensional lattice with local but with no global (average) bias. Related models with global bias have been considered too [8]. In the Sinai model the bias is introduced by asymmetrical transition probabilities and in this way, the bias is coupled with the disorder. An in-depth discussion of another model in discrete time is found in Ref. [7]. Also, a method for calculating the MFPT of a biased random walk on complex networks is reported in Ref. [9]. A related model was treated in Ref. [10], but these authors do not eliminate the stochastic paths reentering into the finite domain of interest from the outside, therefore their results are not correct even for the unbiased case [11].

We introduce a method of working which can be used for a wide kind of biased disordered models. This method allows us to obtain an equation for the MFPT which incorporates in an exact way the effects of disorder. Our description of biased random walk dynamics, on a disordered chain, is made in the framework of the master equation formalism [3, 1]. Let $w_n^{+\,(-)}$ be the transition probability per unit time from site n to $n+1$ ($n-1$). We take

$$w_n^+ = a + \xi_n, \quad w_n^- = b + \xi_n, \qquad (2)$$

where a and b are constants and $\{\xi_n\}$ are taken to be independent but identically distributed random variables with $\langle \xi_n \rangle = 0$. The bias is characterized by the parameter ϵ introduced by $b/a = 1-\epsilon$. We choose $0 \le \epsilon \le 1$, so that the bias field points to the right. The survival probability $F_n(t)$ at time t in the finite interval $D = [-L, L]$ with the initial condition at site n fulfills the backwards master equation. In Laplace representation ($t \to s$), it can be written

$$sF_n(s) - 1 = \left[\mathcal{K}^b + \xi_n \mathcal{K}^0\right] F_n(s) \ , \tag{3}$$

where

$$\begin{aligned} \mathcal{K}^0 &\equiv \mathcal{E}^+ + \mathcal{E}^- - 2\,\mathcal{I}, \\ \mathcal{K}^b &\equiv a\,(\mathcal{E}^+ - \mathcal{I}) + b\,(\mathcal{E}^- - \mathcal{I}), \end{aligned} \tag{4}$$

and $\mathcal{E}^\pm$ are shifting operators $\mathcal{E}^\pm F_n \equiv F_{n\pm1}$ and $\mathcal{I}$ is the identity operator. Equation (3) must be solved with the boundary conditions:

$$F_{-(L+1)}(s) = F_{(L+1)}(s) = 0, \ \ \forall s \tag{5}$$

in order to prevent contributions from trajectories returning to the interval of interest after having left it [11]. The MFPT T_n for leaving the domain D, with initial condition at site n, is obtained from $F_n(s)$ as $T_n = \lim_{s\to0} F_n(s)$.

In a nondisordered chain [$sF_n(s) - 1 = \mathcal{K}^b F_n(s)$], the crossover from the drift (strong bias) regime to the diffusive (weak bias) regime is obtained from the exact solution:

$$T_n = \frac{L+1-n}{a\,\epsilon} - \frac{2(L+1)}{a\,\epsilon}\,\frac{(1-\epsilon)^n - (1-\epsilon)^{(L+1)}}{(1-\epsilon)^{-(L+1)} - (1-\epsilon)^{(L+1)}} \ . \tag{6}$$

Figure 1 shows the behavior of T_n for some values of ϵ. For the directed model ($\epsilon = 1$) we immediately obtain the drift-dominated regime: $T_n = (L+1-n)/a$. In the small bias limit, i.e., $\epsilon \ll 1$, diffusive behavior results

$$T_n \simeq \frac{(L+1)^2 - n^2}{2a}\left(1 - \frac{2n-3}{6}\,\epsilon\right) \ . \tag{7}$$

It is worthwhile to emphasize that the existence of a diffusive behavior is a consequence of the finite size of the domain. For biased RW on a semi-infinite chain ($-\infty$, L], starting from $n \le L$, we obtain the linear relation:

$$T_n = \frac{L+1-n}{a\epsilon}$$

for all values of ϵ. These remarks show that the bias introduces a nontrivial nonlinear dependence of the MFPT in the bias parameter.

In what follows we will study the MFPT in a disordered chain in the presence of a small bias. For weak disorder, from Eq. (2), all the inverse moments

$$\beta_k \equiv \left\langle \left(\frac{1}{a+\xi_n}\right)^k \right\rangle \tag{8}$$

are finite. The positive quantity FQD is given by Eq. (1) using Eq. (8). For finite values of L, Eq. (3) with the prescription given by Eq. (5) is a set of linear algebraic equations that can be exactly solved for $F_n(s)$ and then averaged over the distribution of disorder. If the domain D has a small number of sites N ($= 2L+1$) the problem of solving Eq. (3) reduces to the inversion of an $N \times N$ matrix, and the exact average solution can be obtained by simple configuration counting for disorder with a discrete distribution. For example, we consider $L = 1$ and $\{\xi_n\}$ characterized by

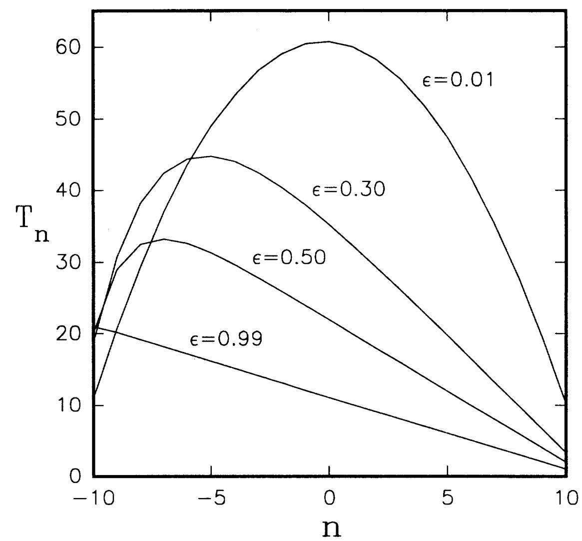

FIG. 1. MFPT for a nondisordered chain [as given by Eq. (6)] plotted against the discrete initial condition n, with $a = 1$ and $L = 10$. The curves shown are only to lead the eye.

$$\{\xi_n\} = \begin{cases} -\frac{1}{2} \text{ with probability } \frac{1}{2} \\ \frac{1}{2} \text{ with probability } \frac{1}{2} \end{cases} \tag{9}$$

so the inverse moments given by Eq. (8) are finite. The number of possible configurations of the disorder is 8, and taking the limit $s \to 0$ we have calculated the disorder-averaged MFPT $\langle T_n(\epsilon)\rangle$ for the different initial conditions $n = -1, 0, 1$. We would like to stress that the range of ϵ is $0 \le \epsilon \le \frac{1}{2}$ in order to satisfy the positivity condition $w_n^- \ge 0$. Our results are shown in the solid lines of Fig. 2 where an unexpected behavior in ϵ is observed. For small bias the mean time needed to leave the interval D results in being larger than the mean exit time in the unbiased case ($\epsilon = 0$). On the other hand, for large bias we get the expected results for a bias pointing to the right: For $n = 1$, the walker starts next to the right frontier so the MFPT becomes smaller than in the unbiased case; for $n = 0$ a similar behavior occurs; but if the walker starts next to the left frontier the bias increases the exit time.

We now show that these numerical results are not a particularity of the small size chosen ($L = 1$), but reflect general properties of the random biased walk. To this end, and following past experience [12], we present an extension of the finite effective-medium approximation (FEMA) [11] that allows us to incorporate the effects of bias. We rewrite the disorder-averaged version of Eq. (3) as a perturbation expansion around the operator $\mathcal{K}^b$:

$$s\,\langle F_n\rangle - 1 = \mathcal{K}^b\,\langle F_n\rangle + \sum_{p=0}^{\infty}\ \sum_{\substack{m_1\neq n\\ m_2\neq m_1\\ \vdots\\ m_p\neq m_{p-1}}} \langle \psi_n\psi_{m_1}\cdots\psi_{m_p}\rangle_T\ J_{nm_1}J_{m_1m_2}\cdots J_{m_{p-1}m_p}\ \mathcal{K}^0\,\langle F_{m_p}\rangle\ . \tag{10}$$

The indices $\{m_i\}$ take values in D, $\langle\cdots\rangle_T$ denotes Terwiel's cumulants, and $\psi_n(s)$ is a ξ-dependent random operator summing up all the terms containing the diagonal parts of $J_{nm}(s)$. The function $J_{nm}(s)$ is defined in terms of the Green function associated with the operator $(s-\mathcal{K}^b)$ with the boundary conditions given by Eq. (5). The mathematical details of this point are left to be published elsewhere. In the following, we consider the case of small bias. It turns out that only the term with $p=0$ in Eq. (10) contributes to order ϵ. Up to this order, the explicit form of Eq. (10) is

$$s\,\langle F_n(s)\rangle - 1 = \beta_1^{-1}\,\mathcal{K}^0\,\langle F_n(s)\rangle - a\left[\mathcal{F}\,\frac{L+1-n}{2(L+1)}\,\mathcal{K}^0 + (\mathcal{E}^- - \mathcal{I})\right]\epsilon\,\langle F_n(s)\rangle\ . \tag{11}$$

Contributions to order $O(\epsilon^2)$ come from all Terwiel's cumulants appearing in Eq. (10). For $\epsilon=0$, Eq. (11) immediately gives the well known MFPT for the unbiased case:

$$\langle T_n(\epsilon=0)\rangle = \frac{(L+1)^2-n^2}{2\,\beta_1^{-1}}, \tag{12}$$

where we can see that the effect of weak disorder is to introduce an effective time scale through the occurrence of the effective diffusion coefficient β_1^{-1}. In the general case, Eq. (11) is a backwards master equation with linear coefficients and its solution has to satisfy the boundary conditions given in Eq. (5).

Instead of working out Eq. (11), we can do a perturbative analysis around an effective *nonhomogeneous* medium. First of all, we write Eq. (3) adding and subtracting a mean-field term: $\Gamma_m(s,\epsilon)\mathcal{K}^0$, $\Gamma_m(s,\epsilon)$ being a nonhomogeneous effective rate to be determined in a self-consistent way, and m is an arbitrary site index fixed in D. The perturbation expansion around the mean-field medium has the form (10) with the following substitutions:

$$\begin{aligned} a &\longrightarrow \bar a &&= a+\Gamma_m(s,\epsilon),\\ b &\longrightarrow \bar b &&= b+\Gamma_m(s,\epsilon),\\ \xi_n &\longrightarrow \eta_n &&= \xi_n-\Gamma_m(s,\epsilon),\\ \epsilon &\longrightarrow \bar\epsilon &&= a\epsilon/\bar a,\\ b/a &\longrightarrow \bar b/\bar a &&= 1-\bar\epsilon. \end{aligned} \tag{13}$$

Now the FEMA consists in truncating the expansion to the first term:

$$s\,\langle F_n(s)\rangle - 1 = \mathcal{K}^b(\bar a,\bar b)\,\langle F_n(s)\rangle \tag{14}$$

and taking $\Gamma_m(s,\epsilon)$ as the solution of $\langle\psi_m(s,\Gamma_m(s,\epsilon))\rangle = 0$. A nontrivial fact results at this point: If the arbitrary site m is taken equal to the initial condition, $m=n$, then Eq. (14) up to order ϵ results in being identical to Eq. (11). Then the FEMA is exact up to this order. The solution of Eq. (14), and then of Eq. (11), is easily obtained by noting that it is the nondisordered result given by Eq. (7) with the substitutions (13). Up to order ϵ and s^0, we obtain for $\Gamma_n(s,\epsilon)$:

$$\bar a = a+\Gamma_n(s,\epsilon) \simeq \beta_1^{-1} - \mathcal{F}\,\frac{L+1-n}{2(L+1)}\,a\,\epsilon, \tag{15}$$

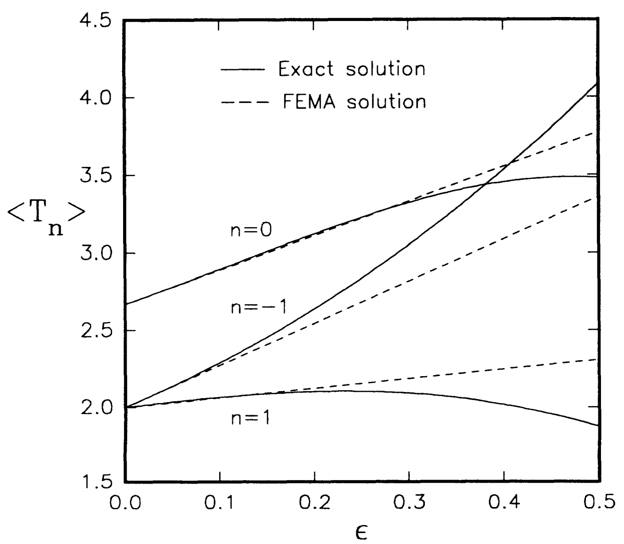

FIG. 2. Disorder-averaged MFPT plotted as a function of the bias parameter for $a=1$ and $L=1$. The curves shown are for the different initial conditions n and correspond to $\mathcal{F}=\frac{1}{4}$. The FEMA solution is a small-ϵ expansion.

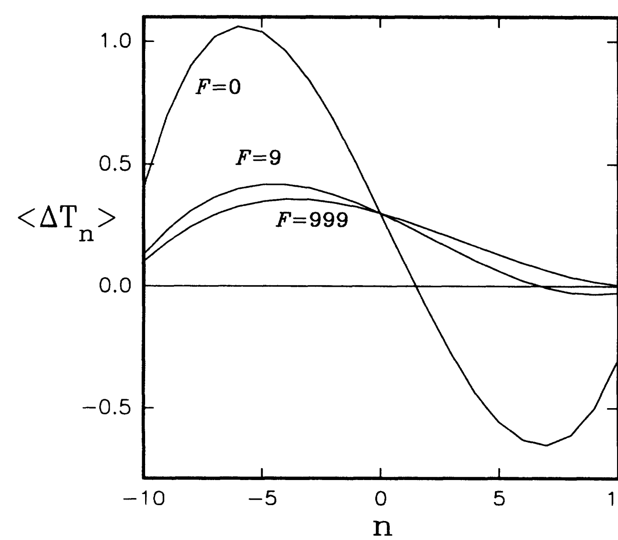

FIG. 3. Plot of the shift $\langle\Delta T_n\rangle$ [as given in Eq. (17)] as a function of the discrete initial condition n for three values of $\mathcal{F}$, with $L=10$ and $E(\mathcal{F}+1)=\frac{1}{100}$. The value $\mathcal{F}=0$ corresponds to the nondisordered case.

which is valid under the following condition for the bias:

$$E \equiv a\beta_1\epsilon \ll \frac{\beta_1^2}{\beta_2} \ . \tag{16}$$

Equation (15) reveals the coupling of the bias to the FQD even to order s^0. To compare with the unbiased case see Eq. (4.21) in Ref. [11]. The comparison between the exact average MFPT with the FEMA approach for the example defined in Eq. (9) is shown in Fig. 2. As expected, we see a good fit for small bias. Finally, to emphasize the effects of bias and the FQD we can write an expression for the averaged shift of the MFPT $\Delta T_n \equiv \beta_1^{-1}\,[T_n - T_n(\epsilon = 0)]$. From Eqs. (7), (12), and (15) we get

$$\langle \Delta T_n \rangle \simeq -\frac{(L+1)^2 - n^2}{2} \left(\frac{2n-3}{6} - \frac{L+1-n}{2(L+1)}\,\mathcal{F} \right) E \ . \tag{17}$$

In Fig. 3 we plot this shift as a function of the initial position n for several values of the FQD. Let us now briefly draw the main conclusion from this result. If the FQD is large enough so that

$$\mathcal{F} \geq \frac{1}{3}\,(2L-3)\,(L+1) \tag{18}$$

the shift is always positive for all initial conditions n. This means that in the presence of bias FQD increases the escape time from the finite interval D. This result raises the interesting experimental possibility of measuring the fluctuation in quenched disordered media.

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